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A Monte Carlo Evaluation of a Nonparametric Technique for Estimating the Hazard Function

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A MONTE CARLO EVALUATION OF A NONPARAMETRIC TECHNIQUE FOR ESTIMATING THE HAZARD FUNCTION

by

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CHAPTER I

INTRODUCTION

This research is primarily concerned with the estimation of the Hazard functions, the Hazard function is the failure rate at time t, and is defined as $-R'(t)/R(t)$, so it plays an important role in Reliability.

In order to compare and evaluate the estimation methods, it is convenient to select one distribution in this research. Since the Weibull distribution is a useful distribution in Reliability, the Weibull distribution is used in this paper.

The Weibull distribution function is given by

$$R(t) = \exp\left[-\left(\frac{t}{Q}\right)^{\alpha}\right] \qquad t \geq o$$

and the Hazard function is given by

$$H(t) = \alpha\, t^{\alpha-1} / Q^{\alpha} \qquad t \geq o.$$

Nonparametric techniques are the primary methods used to estimate the Hazard functions in this paper. The parametric methods given by Bain and Antle (1967) are used to estimate the parameters Q and α in Weibull distribution.

Monte Carlo methods were used to determine the variances and biases of the estimators obtained by both nonparametric and parametric methods so as to evaluate the accuracy and properties of estimators generated by the nonparametric techniques.

The nonparametric estimators were found to have a smaller bias than the parametric method, but the variance is a little larger than the parametric method. When the parametric form of a distribution is unknown the nonparametric method seems to give sufficiently good results to be useful. The nice property of simple computations makes the nonparametric method worthwhile.

CHAPTER II

RELIABILITY AND HAZARD FUNCTION

Reliability (R(t)) of a component is the probability of performing successfully for a specified time t. Experience shows that even a well designed, well engineered, thoroughly tested and properly maintained equipment does not completely avoid the occurence of failures.

The measure of an equipment's reliability is the frequency at which failures occur in time. If there is no failure, the equipment is one hundred percent reliable, if the failure frequency is very low, the equipment's reliability is usually still acceptable. If the failure frequency is high, the equipment is unreliable.

In Engineering and Statistics, reliability has an exact meaning, it can be exactly defined and calculated. The definition is "Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered" (Bazovsky 1962). In the simplest form, Reliability is a probability of success, also referred to as the probability of survival.

The Hazard function (H(t)) is defined as the number of items failing in a time interval divided by the total number of components living at the beginning of this interval, so it can be written as $(f(t)/(1 - F(t)))$, or $-R'(t)/R(t)$. In other words, it is the failure rate at time t as a function of time t.

In order to derive the Reliability and Hazard functions, we assume a fixed number N of components are repeatedly tested, there will be, after a time t, N_s components which survive the test and N_f components which fail,

$$R(t) = N_s/N = 1 - N_f/N$$

$N = N_s + N_f$

$$\frac{dR}{dt} = -\frac{1}{N}\,\frac{dN_f}{dt}$$

$$\frac{-N}{N_s}\,\frac{dR}{dt} = \frac{1}{N_s}\,\frac{dN_f}{dt} = -\frac{1}{R}\,\frac{dR}{dt} = H(t)$$

$$\frac{dR}{R} = -H\,dt \; ; \; \log R = -\int_0^t H\,dt$$

$$R(t) = \exp\left[-\int_0^t H\,dt\right]$$

For the Weibull distribution it is given by

$$R(t) = \exp\left[-\left(\frac{t}{Q}\right)^{\alpha}\right]$$

$$H(t) = -R'(t)/R(t)$$

$$= \alpha\, t^{\alpha-1}/Q^{\alpha}$$

Examples of reliability graphs for the Weibull are seen in Figure 1.

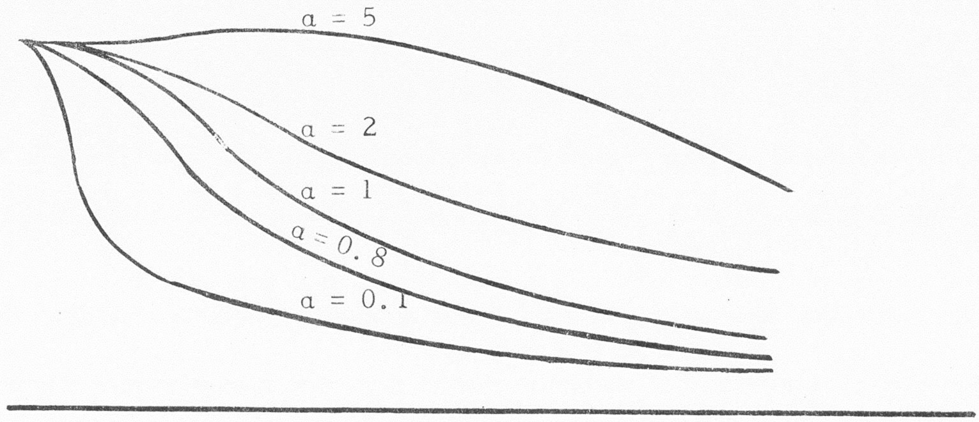

Figure 1. Reliability functions

The graphs of the Hazard functions are found in Figure 2.

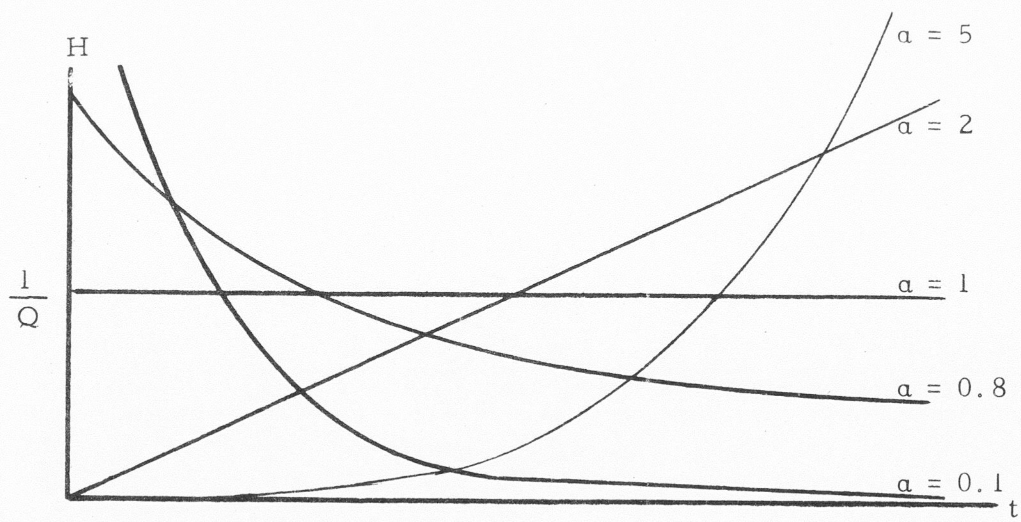

Figure 2. Hazard functions

In many components the hazard function is more complicated. Three kinds of failures may be inherent in equipment, the first is: Early Failures resulting from poor manufacturing and quality control during the production process; the second is Wearout Failures caused by wearout of parts due to improperly maintained; the last is Chance Failures caused by sudden stress accumulations beyond the design strength of the component; failure occurs at random intervals, irregularly and unexpectedly (Bazovsky 1962).

The Early Failure results from poor manufacturing and quality control, so the failures always concentrate in the early time. When substandard components are present in the initial stages of operation, reliability curve declines fast, but it improves rapidly as the failed weak components are replaced by good ones; the reliability curve declines slowly after this period. The curve of the hazard function is similar to the reliability curve. The equipment has a high failure rate in the initial period and declines after the early failures have occurred.

The figures and formulas of Early Failures are found in Figure 3, 4.

The Chance Failures caused by sudden stress accumulations, occurs at random intervals, irregularly and unexpectedly. So the Failures distribute evenly over the whole life time of equipment, and the Failure rate is a constant during the life time.

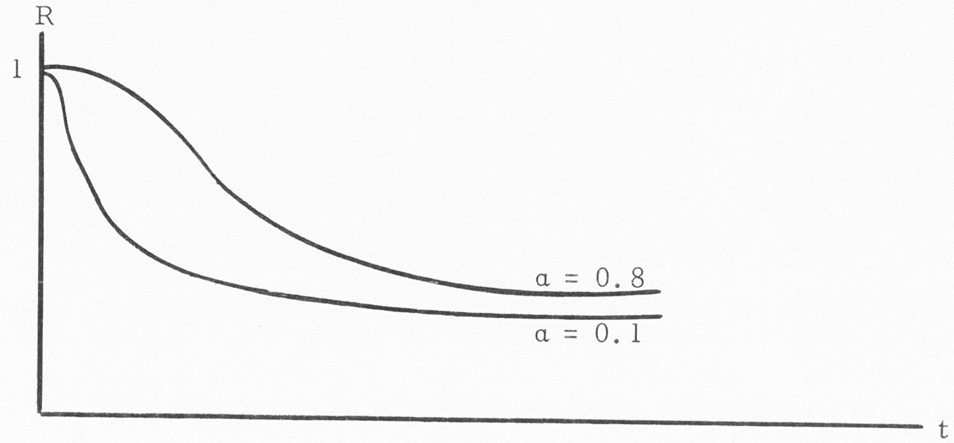

Figure 3. Reliability function of Early Failures

$$R_1(t) = \exp\left[-\left(\frac{t}{Q_1}\right)^{a_1}\right] \qquad a_1 < 1$$

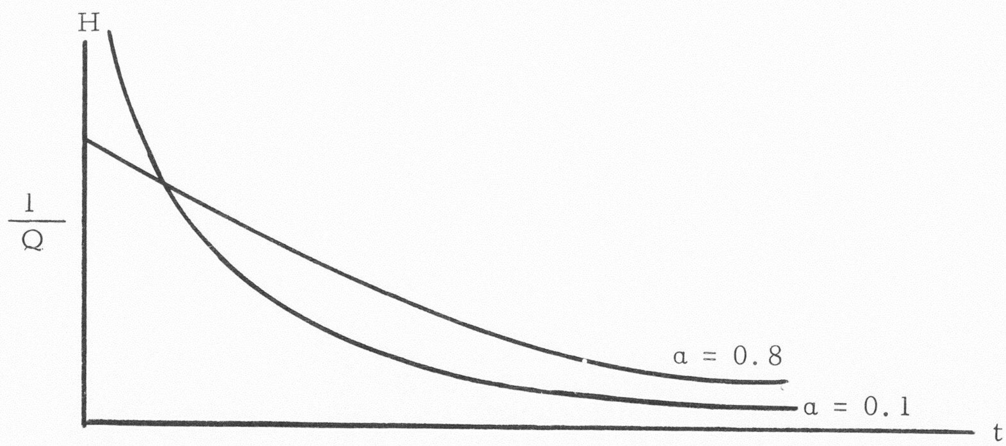

Figure 4. Hazard function of Early Failures

$$H_1(t) = a_1 t^{a_1 - 1} / Q_1^{a_1} \qquad a_1 < 1$$

We get a smooth descending curve of Reliability, and a constant Hazard function as shown in Figure 5 and 6.

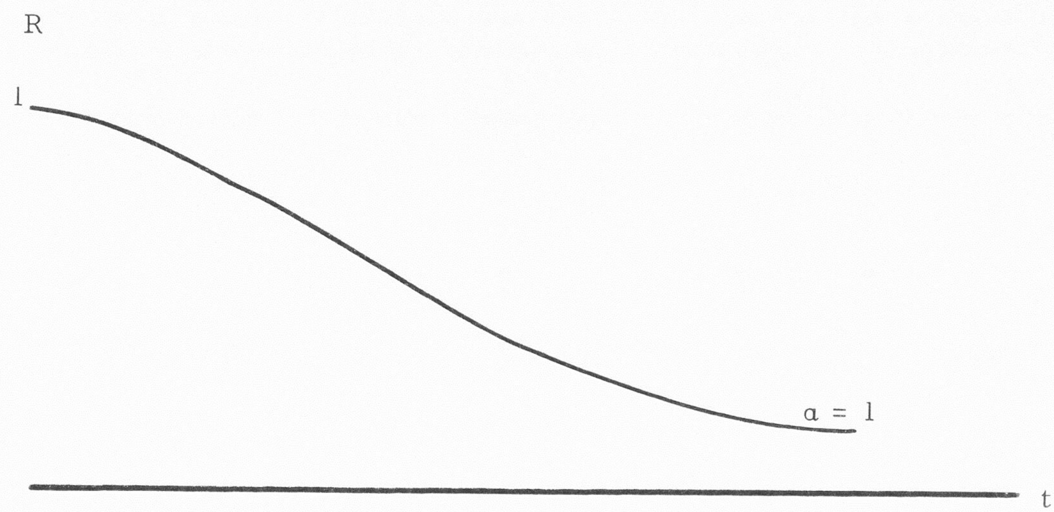

Figure 5. Reliability function of Chance Failures

$$R_2(t) = \exp\left[-\left(\frac{t}{Q_2}\right)^{a_2}\right] \qquad a_2 = 1$$

$$= \exp\left[-\left(\frac{t}{Q_2}\right)\right]$$

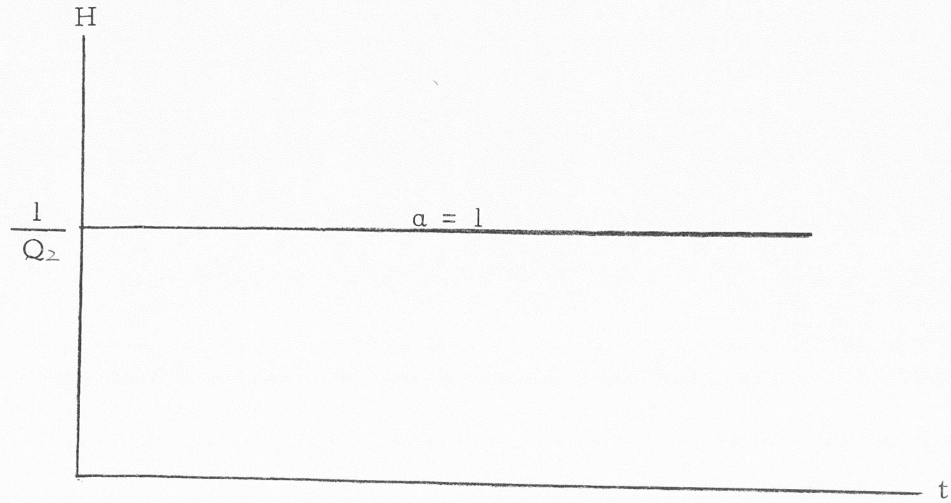

Figure 6. Hazard function of Chance Failures

$$H_2(t) = \alpha_2 t^{\alpha_2 - 1} / Q_2^{\alpha_2} \qquad \alpha_2 = 1$$

$$= \frac{1}{Q_2}$$

The Wearout Failures are caused by wearout of parts due to improper maintance or exhaustion. Sometimes they are approximated by a normal distribution; about one-half of failures occur before the mean life time M, and one-half occur later. Example of density function of the Wearout Failures is seen in Figure 7.

$$f(t) = \frac{\qquad}{\sigma \sqrt{2\eta}} \; e^{-(t - M) / = \sigma^2}$$

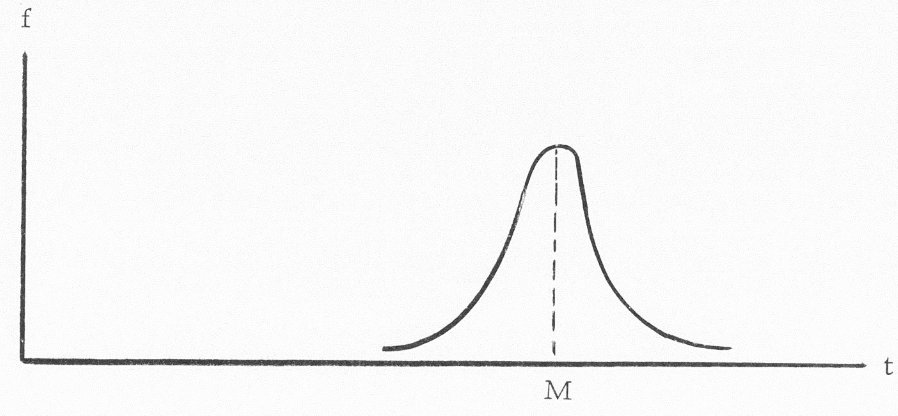

Figure 7. Density function of the Wearout Failures

The Reliability curve is found in Figure 8.

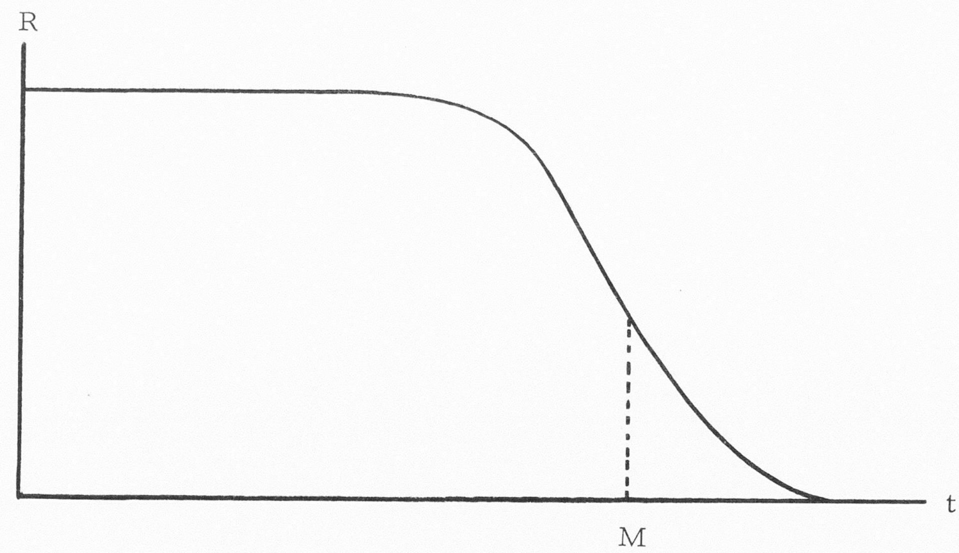

Figure 8. Reliability function of Wearout Failures

$$R_3(t) = 1 - \int_0^t f(t)\,dt = \int_t^\infty f(t)\,dt$$

$$H_3(t) = f(t) \,/\, R(t)$$

$$= f(t) \,/ \int_t^\infty f(t)\,dt$$

But in most cases, the equipment suffers its greatest Wearout Failures in the test period before M when components are allowed to wearout and are replaced only as they fail (Bazovsky 1962). So the Wearout Failures always follow the Weibull distribution.

The reliability curve of Wearout Failures shows almost no decline in the initial period and begins to decline at some place before M. The Reliability curves are seen in Figure 9.

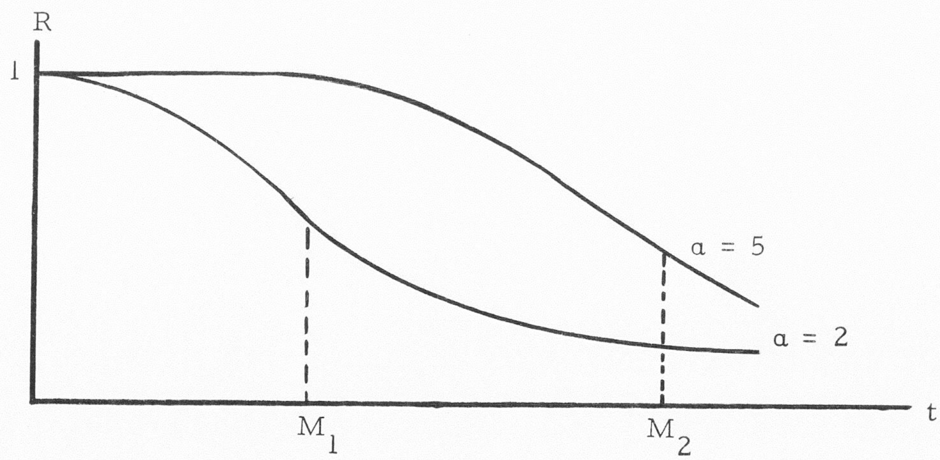

Figure 9. Reliability function of Wearout Failures

$$R_3(t) = \exp \left[-\left(\frac{t}{Q_3}\right)^{a_3} \right] \qquad a_3 > 1$$

The Hazard function is near zero initially. It ascends gradually when time t is getting larger, since the failure rate is the increasing function of time t. The Hazard function is seen in Figure 10.

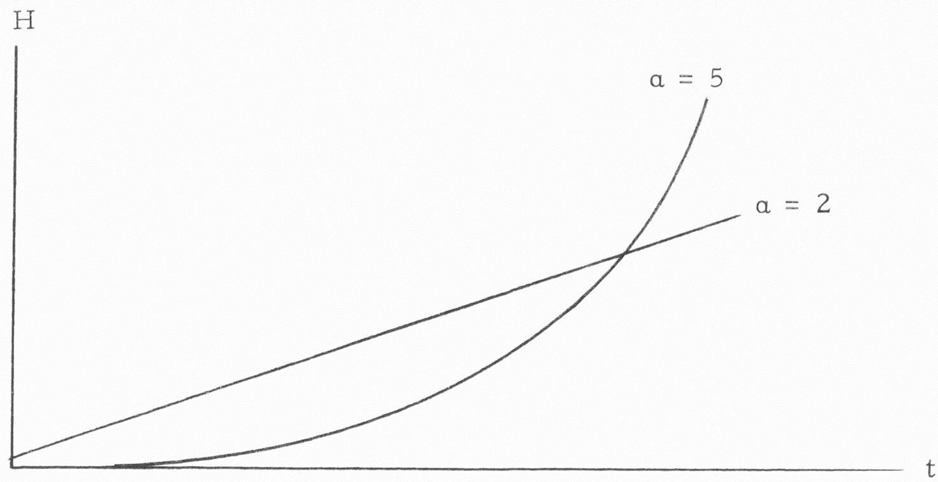

Figure 10. Hazard function of Wearout Failures

$$H_3(t) = a_3 \, t^{a_3 - 1} \big/ Q_3^{a_3} \qquad a_3 > 1$$

The combined Reliability function of an equipment is given

$$R(t) = R_1(t)\ R_2(t) - R_3(t)$$

$$= \exp\ - \left\{ \left[: (\frac{t}{Q_1})^{a_1} + (\frac{t}{Q_2}) + (\frac{t}{Q_3})^{a_3} \right] \right\}$$

When t is small, the term $(t/Q_3)^{a_3}$ is usually neglected; it means the Wearout Failures are not obvious. When t is larger, the term $(t/Q_3)^{a_3}$ plays an important role in R(t); Early Failures are usually neglected.

We can draw the curve of combined Hazard function approximately (Bazovsky 1962) as shown in Figure 11.

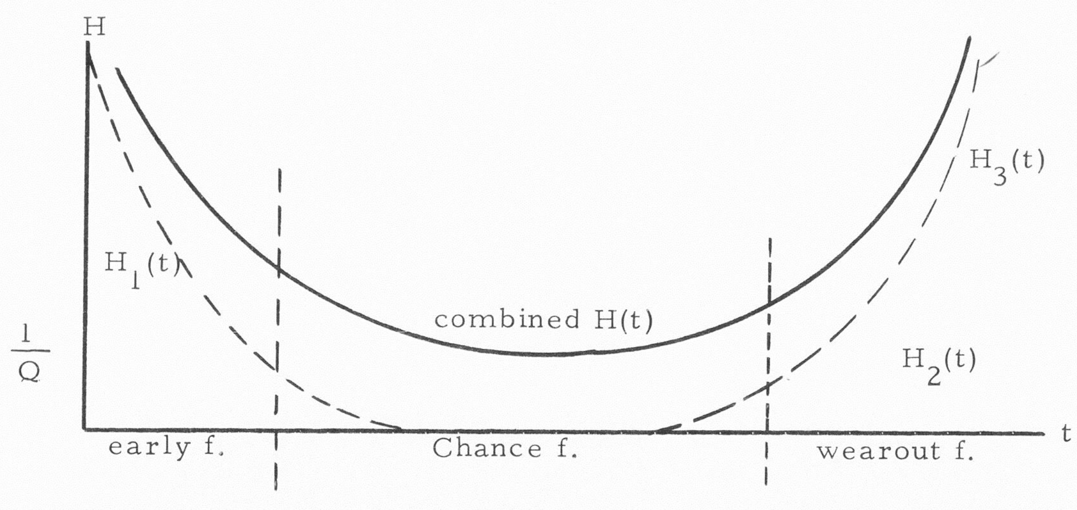

Figure 11. Combined hazard function

$$H(t) = a_1 t^{a_1 - 1} / Q_1^{a_1} + 1/Q_2 + a_3 t^{a_3 - 1} / Q_3^{a_3}$$

CHAPTER III

ESTIMATION OF HAZARD FUNCTION

From Chapter II, it is noted that the hazard function may be a complex function which combines more than one different shapes of curves with different parameters. Thus it may be very difficult to estimate the combined Hazard function. In applications, attention is usually focused on a portion of the Hazard function, such as the Hazard function of early failures, or wearout failures, but not the whole combined hazard function.

Hazard function is defined as $(f(t)/(1-F(t)))$, where $F(t)$ is the failure distribution function, and $f(t)$ is the density function. N samples of time to failure are obtained from the population.

According to the nonparametric techniques, the N samples of failure times t t divide the whole population into N+1 equal portions,

let t_1 t_2 : t_i t_N be N failure times

d_1 d_2 · d_i d_N be the time interval between successive samples, so $d_i = t_{i+1} - t_i$. Then

$f(t)$ is the slope of the interval d_i, and

$F(t)$ is the cumulative probability at t_i.

From Figure 12,

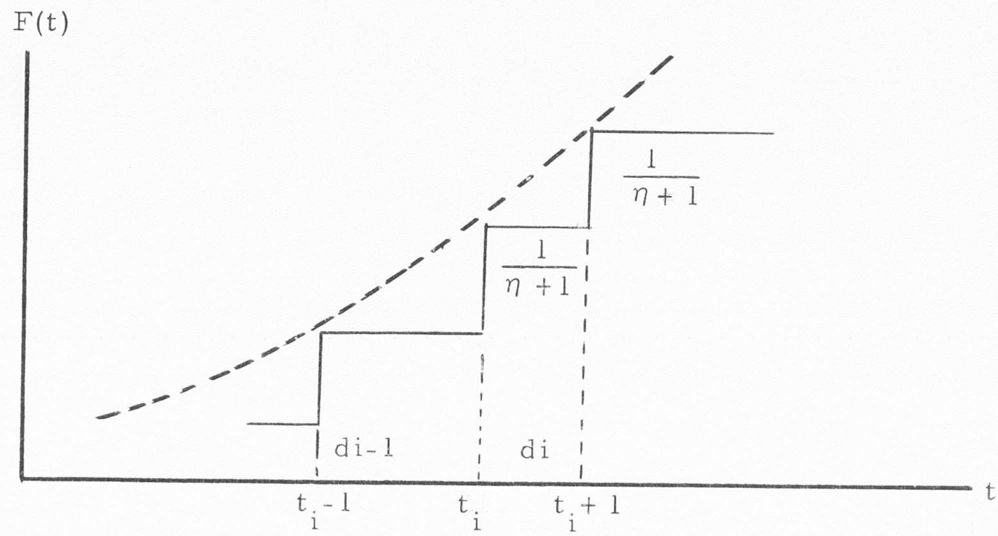

Figure 12. Failures distribution function

we get

$$f(t_i) = \frac{\frac{1}{\eta+1}}{di} = \frac{1}{(\eta+1)\, di}$$

$$F(t_i) = \frac{i}{\eta+1}$$

3.1 First Method

To estimate Hazard function at time T, (H(T)), we must estimate the f(T) and F(T) first,

$$F(T) = \frac{i}{\eta+1} + \left(\frac{T - t_i}{di}\right) \Big/ (\eta+1)$$

$$= \frac{1}{\eta+1} \quad \left(i + \frac{T - t_i}{di}\right)$$

d_i is the interval of $t_{i+1} - t_i$ where T is located.

The f(T) is the slope of F(t) at T, since the curve of F(t) is assumed to be very smooth, if f(T) is estimated using only one interval, the variance of the estimators will be larger, so several intervals around T must be counted, and each slope is weighed by the length of interval itself, this is comparable with moving averages used in time series. During the research, 3, 5, 9, 12, and 15 intervals have been tried, 9 intervals is found to give good results when sample size N is equal to 100.

$$f(T) = \frac{\left[\frac{1}{(\eta+1)d_{i-4}} \times d_{i-4} + \cdots + \frac{1}{(\eta+1)\, d_{i+4}} \times d_{i+4}\right]}{(d_{1-4} + \cdots + d_i + \cdots + d_{i+4})}$$

$$= \frac{\frac{9}{n+1}}{t_{i+5} - t_{i-4}}$$

$$= \frac{9}{(\eta+1)\,(t_{i+5} - t_{i-4})}$$

$$\hat{H}(T) = f(T) \Big/ (1 - F(T))$$

In order to compare and evaluate this method with a parametric method, the Weibull distribution is used in a monte carlo study, the results of this study are found in Chapter V, and computer program is found in Appendix A.

3.2 Second Method

In this method, a new conception of hazard function is considered, as shown in Figure 13,

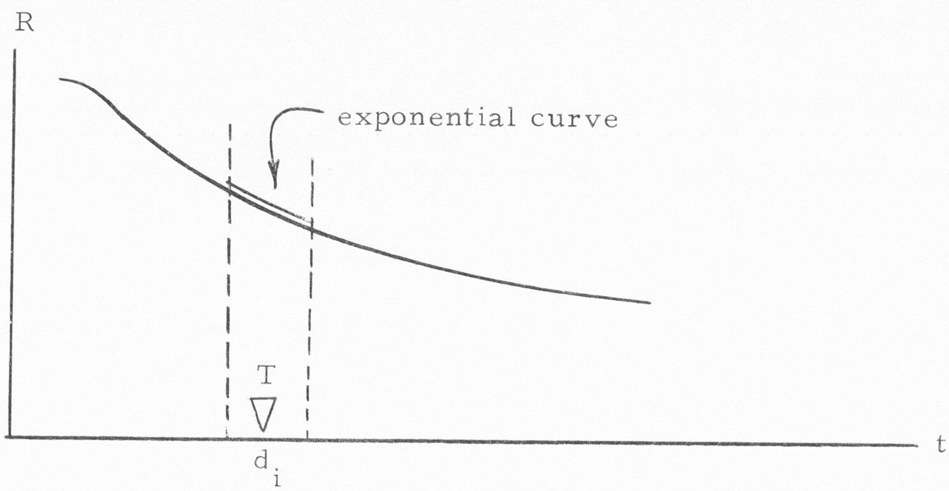

Figure 13. Weibull treated as an exponential

the reliability curve is a Weibull distribution, but we can treat the reliability curve as an Exponential distribution ($R(t) = e^{-Ht}$) in a short interval d_i, where the T is located. Then the hazard function ($H(T)$) may be assumed to be a constant in exponential distribution.

For the exponential distribution $R(t) = e^{-Ht}$, H is the hazard function and is the reciprocal of the mean m.

$$H = \frac{1}{m}$$

During this research, 9 intervals around T is found to be a good interval for estimating the hazard function when sample size is 100.

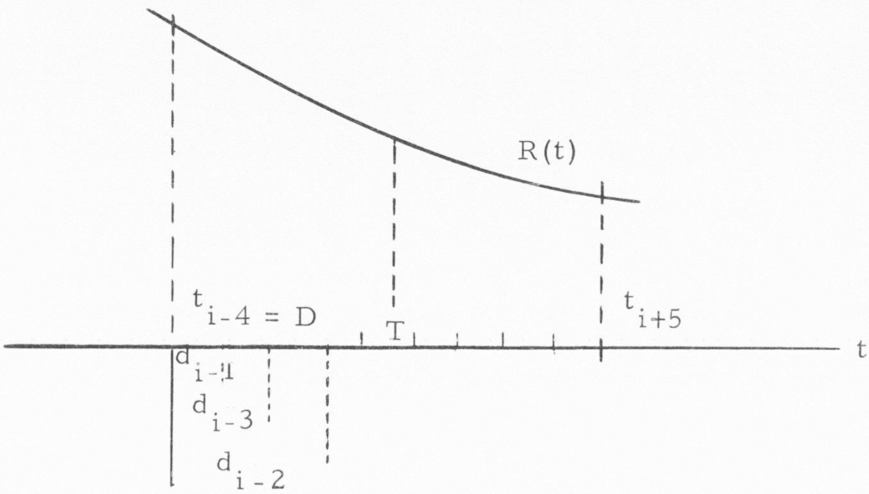

Figure 14. Enlarged figure of the short interval of exponential curve from Figure 13.

From Figure 14, let d_i-4 ∘ d_i ∘ ∘ ∘ d_{i+4} be the 9 intervals included in the computation, and let T be located in interval d_i, $t_{i-4} = 0$ be the beginning of the exponential curve,

$M = \dfrac{\Sigma X}{\eta}$ X is the variables of t in exponential distribution,

$X_1 = d_{i-4} = t_{i-3} - t_i - 4.$

$X_2 = d_{i-3} = t_{i-2} - t_{i-4}$

$$X_9 = d_{i+4} = t_{i+5} - t_{i-4}$$

$$\eta = 9$$

$$\hat{H}(T) = \frac{1}{M} = \frac{\eta}{\Sigma X}$$

$$= \frac{9}{X_1 + X_2 + \cdots + X_9}$$

In order to compare and evaluate this method with a parametric method, the Weibull distribution is used in a monte carlo study, the results of this study are found in Chapter V, and computer program is found in Appendix B.

CHAPTER IV

ESTIMATION WITH THE PARAMETRIC TECHNIQUES

In order to compare and evaluate how good the nonparametric technique is, the parametric technique given by Bain and Antle (1967) is used to estimate the parameter Q and α in the Weibull distribution.

For the Weibull, let

$$\Xi(t_1 \alpha_1 Q) = (t/Q)^{\alpha}$$

then

$$f(\Xi) = e^{-\Xi} \qquad \Xi \geq 0$$

$$E(\Xi_{ixi}) = S_i = \sum_{i=1}^{i} \frac{1}{\eta - i + 1}$$

(Bain and Antle, 1967)

The least square method should minimize $\sum_{i=1}^{\eta} \left[(t_{(i)}/Q)^{\alpha} - S_i \right]^2$

A simple method is to minimize $\sum_{i=1}^{\eta} \left[\text{LOG } (t_{(i)}/Q)^{\alpha} - \text{LOG } S. \right]^2$

Let

$$Y = \mathop{S}_{x=1}^{\eta} \left[\text{LOG } (t_{(i)}/Q)^{\alpha} - \text{LOG } S_i \right]^2$$

$$= \underset{i=1}{\overset{\eta}{S}} \left[a \text{ LOG } t(i) - a \text{ LOG } Q - \text{LOG } S_i \right]^2$$

$$= \underset{i=1}{\overset{\eta}{S}} \left[a^2 \cdot (\text{LOG } t_{(i)}) + a^2 (\text{LOG } Q)^2 + (\text{LOG } S_i)^2 \right.$$

$$- 2a^2 \text{ LOG } t_{(i)} \text{ LOG } Q + 2a \text{ LOG } Q \text{ LOG } S_i$$

$$\left. - 2a^2 \text{ LOG } t_{(i)} \cdot \text{LOG } S_i \right]$$

To find the minimum of y, let

$\frac{ay}{aQ} = 0$ we get

$$2\eta a^2 \cdot \frac{1}{Q} \text{LOG } Q - 2a^2 \cdot \frac{1}{Q} (S \text{ LOG } + t(2)) + 2a \cdot \frac{1}{Q} \cdot \Sigma \text{ LOG } S_i = 0$$

$$\eta a \text{ LOG } Q = a \text{ LOG } \pi t(i) - \Sigma \text{ LOG } S_i$$

$$Q^{\eta a} = \left[\pi t(i) \right]^a / \pi S_i$$

$$\hat{Q} = \left[\pi t(i) \right]^{\frac{1}{\eta}} / \left[\pi Si \right]^{\frac{1}{\eta a}}$$

$\frac{ay}{2a} = 0$ we get

$$Za\Sigma(\text{LOG } t(i)^2 + 211a (\text{LOG } Q)^2 - 4a \text{ LOG } Q \cdot (\Sigma \text{ LOG}, t(i))$$

$$+ Z \text{ LOG } Q \text{ S LOG } S_i - 2\Sigma (\text{LOG } t(i) \cdot \text{LOG } S_i) = 0$$

$$\alpha\Sigma(\mathrm{LOG}\ t(i))^2 - \frac{\alpha}{11}(\Sigma\ \mathrm{LOG}\ t(i))^> \frac{1}{11}\ S\ \mathrm{LOG}\ t(i) \cdot \Sigma\ \mathrm{LOG}\ S_i$$

$$- \Sigma\ (\mathrm{LOG}\ t(i) \cdot \mathrm{LOG}\ S_i) = 0$$

$$\alpha\left[\Sigma(\mathrm{LOG}\ t(i))^2 - \frac{1}{11}(\Sigma\ \mathrm{LOG}\ t(i))^2\right] = \Sigma(\mathrm{LOG}\ t(i) \cdot \mathrm{LOG}\ S_i)$$

$$- \frac{1}{\eta}\ \Sigma\ \mathrm{LOG}\ t(i) \cdot \Sigma\ \mathrm{LOG}\ S_i$$

$$\hat{\alpha} = \frac{\Sigma(\mathrm{LOG}\ t(i) \cdot \mathrm{LOG}\ S_i) - \frac{1}{\eta}\ \Sigma\ \mathrm{LOG}\ t(i) \cdot S\ \mathrm{LOG}\ S_i}{S\ (\mathrm{LOG}\ t(i))^2 - \frac{1}{\eta}(\Sigma\ \mathrm{LOG}\ t(i))^2}$$

the estimator of the hazard function is given by

$$\hat{H}(t) = \hat{\alpha} \cdot t^{\hat{\alpha}-1} / \hat{Q}^{\hat{\alpha}}$$

In practice, it is usually necessary to estimate the hazard function from truncated samples; that is, it is usually not possible to continue a testing program until all samples on test have failed. The nonparametric methods are not affected by truncation; however, the parametric techniques are affected by truncation.

It was arbitrarily assumed that the samples will be truncated after t(70), where t(70) is the 70th smallest observation. If the truncation is carried out before t(70), then the variance of the parametric estimator will be larger. The variance of the nonparametric estimator will remain the same in this case; however, the range over which the hazard function may be estimated becomes smaller.

In order to compare and evaluate with nonparametric method, the Weibull distribution is used in a monte carlo study; the results of this study are found in Chapter V, and computer program is found in Appendix C.

CHAPTER V

EVALUATION AND SUMMARY

In the monte carlo simulation, a Weibull distribution was assumed, and the values of parameters were selected as;

$Q = 10$

$\alpha = 0.4,\ 0.8,\ 2.$

From each sample of size 100, the hazard function was estimated both by parametric and nonparametric techniques at the following 3 times, $T = Q/5,\ Q/2.\ Q$. These estimates were repeated for 500 times and the mean and variance of the estimators were calculated. The results of this simulation are recorded in Table 1.

The estimators from the second nonparametric method are found to have a large variance and bias; therefore, it is not a good method for estimating, so in this chapter, only the first nonparametric method is compared and evaluated.

From Table 1, it is found that the nonparametric estimators have the following characteristics;

1. It has a smaller bias than the parametric method.

2. The Coefficient of variation is almost the same in parametric method. But in the parametric method, it goes up when T becomes larger.

3. The nonparametric method is not influenced by truncated samples while parametric method is affected by truncation.

4. It has a little larger variances than the parametric method, but it is not too large to be unacceptable, especially for large times. When the parametric form is unknown, it seems to give sufficiently good results.

5. It has a nice property of simple computations as noted in Chapter III and IV.

In general, the nonparametric technique may be useful method for estimating the hazard function when parametric techniques are not available.

Table 1. Comparison of Bias and Variance for estimation of Hazard function

Q = 10		T = Q/5			T = Q/2			T = Q		
		bias 10−3	var. 10−3	coef. $S/\overline{X}$	bias 10−3	var. 10−3	coef. $S/\overline{X}$	bias 10−3	var. 10−3	coef. $S/\overline{X}$
α=0.4	*N	−1.000	1.448	0.365	−1.300	0.538	0.391	−0.970	0.191	0.353
	*P	−5.200	0.382	0.196	−3.680	0.180	0.235	−2.700	0.101	0.269
α=0.8	N	0.600	1.577	0.358	0.260	1.038	0.349	−1.060	0.738	0.343
	P	−2.500	0.249	0.147	−4.790	0.318	0.205	−5.410	0.404	0.269
α=2	N	−2.170	0.984	0.827	−1.270	1.366	0.374	−6.100	5.222	0.372
	P	4.580	0.121	0.246	−1.730	0.195	0.142	−13.500	2.523	0.270

*N Nonparametric
*P Parametric

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APPENDIXES

Appendix A

$Q = 10$

$\alpha = 0.4,\ 0.8,\ 2$

$T = Q/5,\ Q/2,\ Q$

sample size N = 100, and repeat 500 times in each case.

Computer Program

Main program

```
      DIMENSION T(100), TT(3), DEV(3), SD(3), HT(3), H(500, 3),
               AVE(3)
200   FORMAT(3X, 'HAZARD', /, (3X, 9(2X, E11.4)))
300   FORMAT (3X, 'DEVIATION', /, (3X, 3(2X, E11.4)))
      DATA N, M, THETA, ALPHA/100, 500, 10., α/
      XM=M
      XN=N+1
      AMDA=1./ALPHA
      TT(1)=THETA/5.
      TT(2)=THETA/2.
      TT(3)=THETA
      DO 40 I=1, M
      DO 10 J=1, N
      F=RN(27571)
```

```
10    T(J)=THETA* (ABS(ALOG(F)))**AMDA
      CALL SORT (T, N)
      JJ=1
      NN=N-1
      DO 40 K=1,3
      DO 20 L=JJ, NN
      IF(TT(K).GT.T(L)) GO TO 20
      JJ=L-1
      X=TT(K)
      GO TO 30
20    CONTINUE
30    CALL HAZARD(JJ, X, XN, T, HA, N)
40    H(I, K)=HA
      WHITE(6, 200) ((H(I, K), K=1,3), I=1, 500)
      DO 60 I=1,3
      HT(I)=ALPHA*TT(I)**(ALPHA-1.))/THETA**ALPHA
      SS1=0.
      SS2=0.
      DEV(I)=0.
      DO 50 J=1, M
      SS1=SS1+H(J, I)*H(J, I)
      SS2=SS2+H(J, I)
```

```
50      DEV(I)=DEV(I)+(ABS(H(J,I)-HT(I)))**2
        DEV(I)=DEV(I)/(XM-1.)
        AVE(I)=SS2/XM
60      SD(I)=(SS1-SS2*SS2/XM)/XM-1.)
        WRITE(6,300)DEV,SD,HT,AVE
        STOP
        END

        SUBROUTINE HAZARD(J,X,S,T,HA,N)
        DIMENSION T(N)
        R=J
        D=T(J+1)-T(J)
        G=(X-T(J))/D
        FS=9./(S*(T(J+5)-T(J-4)))
        FC=(R+G)/S
        HA=FS/(1.-FC)
        RETURN
        END

        SUBROUTINE SORT(X,N)
        DIMENSION X(N)
        M=N
3       M=M/2
        IF(M.EQ.O) GO TO 7
```

```
      K=N-M
      J=1
4     I=J
5     L=I+M
      IF(X(I). LE. X(L)) FO TO 6
      T=X(I)
      X(I)=X(L)
      X(L)=T
      I=I-M
      IF(I. GE. '') GO TO 5
6     J=J+1
      IF(J-K) 4, 4, 3
7     RETURN
      END
```

Appendix B

Q = 10

α = 0.4, 0.8, 2

T = Q/5, Q/2, Q

sample size N = 100, and repeat 500 times in each case.

Computer program

Main program

(the same as in appendix 1)

```
SUBROUTINE HAZARD(J, T, HA, N)
DIMENSION T(N)
R=T(J-4)
S=T(J-3)+T(J-2)+T(J-1)+T(J)+T(J+1)+T(J+2)+T(J+3)+T(J+4)+T(J+5)
HA=9./(S-9*R)
RETURN
END
```

```
SUBROUTINE SORT
```

(the same as in appendix 1)

Appendix C

$Q = 10$

$\alpha = 0.4,\ 0.8,\ 2$

$T = Q/5, Q/2,\ Q$

sample size N = 100, and repeat 500 times in each case.

Computer program

Main program

```
      DIMENSION T(100), ALPHA(500), THETA(500), HT(3), TT(3)
            DEV(3), SD(3), AVE(3), HA(500, 3)
      DATA M, N, XM, TT(1), TT(2), TT(3)/500, 100, 500., 2., 5., 10./
      XN=70.
      P=1./XN
      DO 10 I=1, M
      S1=0.
      S2=1.
      S3=1.
      A=0.
      B1=0.
      B2=0.
      C=0.
```

```
      DO 20 K=1,N
      F=RN(27571)
   20 T(K)= Q * (ABS(ALOG(F)))**(1/a
      CALL SORT (T,N)
      DO 60 J=1,70
      XJ=J
      S1=S1+1./(100.-XJ+1.)
      S2=S2*(T(J))**P
      S3=S3*S1**P
      U=ALOG(T(J))
      V=ALOG(S1)
      A=A+U*V
      B1=B1+U
      B2=B2+V
   60 C=C+(ABS(U))**2
      B=B1*B2/XN
      D=B1*B1/XN
      ALPHA(I)=A-B)/(C-D)
      W=(C-D)/(A-B)
      G=S3**W
      THETA(I)=S2/G
   10 CONTINUE
      WRITE(6,200) ALPHA
```

```
200 FORMAT(5X,'THE ALPHA ESTIMATORS',/,(5X,10(1X,E11.4)))
    WRITE(6,300) THETA
300 FORMAT(5X,'THE THETA ESTIMATORS',/,(5X,10(1X,E11,4)))
    DO 40 J=1,3
    HT(J)=α *TT(J)**(α-1.)/Q**Q
    SS1=0.
    SS2=0.
    DEV(J)=0.
    DO 30 I=1,M
    HA(I,J)=(ALPHA(I)*TT(J)**(ALPHA(1)-1.))/THETA(I)**ALPHA(I)
    SS1=SS1+HA(I,J)*HA(I,J)
    SS2=SS2+HA(I,J)
 30 DEV(J)=DEV(J)+(ABS(HA(I,J)-HT(J)))**2
    DEV(J)=DEV(J)/XM-1.)
    AVE(J)=SS2/XM
 40 SD(J)=(SS1-SS2*SS2/XM)/(XM-1.)
    WRITE(6,400) ((HA(I,J),J=1,3),I=1,500)
400 FORMAT(5X,'THE ESTIMATORS OF HAZARD FUNCTION',/,
    (5X,9(2X,E11.4)))
    WRITE (6,500) DEV,SD,HT,AVE
500 FORMAT(5X,'THE DEVIATIONS',/,(5X,3(2X,E11.4)))
    STOP
    END
```

SUBROUTINE SORT (T,N)

(the same as the appendix 1)